

The Rank of Tree-Automatic Linear Orderings

Martin Huschenbett

Institut für Theoretische Informatik, Technische Universität Ilmenau, Germany
martin.huschenbett@tu-ilmenau.de

Abstract

We generalise Delhommé's result that each tree-automatic ordinal is strictly below ω^{ω^ω} by showing that any tree-automatic linear ordering has FC-rank strictly below ω^ω . We further investigate a restricted form of tree-automaticity and prove that every linear ordering which admits a tree-automatic presentation of branching complexity at most $k \in \mathbb{N}$ has FC-rank strictly below ω^k .

1 Introduction

In [4], Delhommé showed that an ordinal α is string-automatic if, and only if, $\alpha < \omega^\omega$ and it is tree-automatic if, and only if, $\alpha < \omega^{\omega^\omega}$. Khoussainov, Rubin, and Stephan [7] extended his technique to prove that every string-automatic linear ordering has finite FC-rank. Although it is commonly expected that every tree-automatic linear ordering has FC-rank below ω^ω , this conjecture has not been verified yet.¹ We close this gap by providing the missing proof (Theorem 4.4). As part of this, we give a full proof of Delhommé's decomposition theorem for tree-automatic structures (Theorem 3.6). Afterwards, we investigate a restricted form of tree-automaticity where the branching complexity of the trees involved is bounded. We show that each linear ordering which admits a tree-automatic presentation of branching complexity $k \in \mathbb{N}$ has FC-rank below ω^k (Theorem 5.4). As a consequence, we obtain that an ordinal α admits a tree-automatic presentation whose branching complexity is bounded by k if, and only if, $\alpha < \omega^{\omega^k}$.

2 Tree-Automatic Structures

This section recalls the basic notions of tree-automatic structures (cf. [1, 2]).

Let Σ be an alphabet. The set of all (*finite*) words over Σ is denoted by Σ^* and the *empty word* by ε . A *tree domain* is a finite, prefix-closed subset $D \subseteq \{0, 1\}^*$. The *boundary* of D is the set $\partial D = \{ud \mid u \in D, d \in \{0, 1\}, ud \notin D\}$ if D is not empty and $\partial \emptyset = \{\varepsilon\}$ otherwise. A Σ -*tree* (or just *tree*) is a map $t: D \rightarrow \Sigma$ where $\text{dom}(t) = D$ is a tree domain. The *empty tree* is the unique Σ -tree t with $\text{dom}(t) = \emptyset$. The set of all Σ -trees is denoted by T_Σ and its subsets are called (*tree*) *languages*. For $t \in T_\Sigma$ and $u \in \text{dom}(t)$ the *subtree* of t rooted at u is the tree $t|u \in T_\Sigma$ defined by

$$\text{dom}(t|u) = \{v \in \{0, 1\}^* \mid uv \in \text{dom}(t)\} \quad \text{and} \quad (t|u)(v) = t(uv).$$

For $u_1, \dots, u_n \in \text{dom}(t) \cup \partial \text{dom}(t)$ which are mutually no prefixes of each other and trees $t_1, \dots, t_n \in T_\Sigma$ we consider the tree $t[u_1/t_1, \dots, u_n/t_n] \in T_\Sigma$. Intuitively, $t[u_1/t_1, \dots, u_n/t_n]$ is obtained from t by simultaneously replacing for each $i = 1, \dots, n$ the subtree rooted at u_i by t_i . Formally,

$$\text{dom}(t[u_1/t_1, \dots, u_n/t_n]) = \text{dom}(t) \setminus (\{u_1, \dots, u_n\}\{0, 1\}^*) \cup \bigcup_{1 \leq i \leq n} \{u_i\} \text{dom}(t_i)$$

¹ Recently, Jain, Khoussainov, Schlicht, and Stephan [6] independently from us obtained results which verify this conjecture as well.

and

$$(t[u_1/t_1, \dots, u_n/t_n])(u) = \begin{cases} t_i(v) & \text{if } u = u_i v \text{ for some (unique) } i \in \{1, \dots, n\}, \\ t(u) & \text{otherwise.} \end{cases}$$

A (*deterministic bottom-up*) *tree automaton* $\mathcal{A} = (Q, \iota, \delta, F)$ over Σ consists of a finite set Q of *states*, a *start state* $\iota \in Q$, a *transition function* $\delta: \Sigma \times Q \times Q \rightarrow Q$, and a set $F \subseteq Q$ of *accepting states*. For all $t \in T_\Sigma$, $u \in \text{dom}(t) \cup \partial \text{dom}(t)$, and maps $\rho: U \rightarrow Q$ with $U \subseteq \partial \text{dom}(t)$ a state $\mathcal{A}(t, u, \rho) \in Q$ is defined recursively by

$$\mathcal{A}(t, u, \rho) = \begin{cases} \delta(t(u), \mathcal{A}(t, u0, \rho), \mathcal{A}(t, u1, \rho)) & \text{if } u \in \text{dom}(t), \\ \rho(u) & \text{if } u \in U, \\ \iota & \text{if } u \in \partial \text{dom}(t) \setminus U. \end{cases}$$

The second parameter is omitted if $u = \varepsilon$ and the third one if $U = \emptyset$. Notice that $\mathcal{A}(t, u) = \mathcal{A}(t \upharpoonright u)$. The tree language *recognised* by \mathcal{A} is the set

$$L(\mathcal{A}) = \{ t \in T_\Sigma \mid \mathcal{A}(t) \in F \}$$

of all trees which yield an accepting state at their root. A language $L \subseteq T_\Sigma$ is *regular* if it can be recognised by some tree automaton.

Let $\square \notin \Sigma$ be a new symbol and $\Sigma_\square = \Sigma \cup \{\square\}$. The *convolution* of an n -tuple $\bar{t} = (t_1, \dots, t_n) \in (T_\Sigma)^n$ of trees is the tree $\otimes \bar{t} \in T_{\Sigma_\square}$ defined by

$$\text{dom}(\otimes \bar{t}) = \text{dom}(t_1) \cup \dots \cup \text{dom}(t_n) \quad \text{and} \quad (\otimes \bar{t})(u) = (t'_1(u), \dots, t'_n(u)),$$

where $t'_i(u) = t_i(u)$ if $u \in \text{dom}(t_i)$ and $t'_i(u) = \square$ otherwise. A relation $R \subseteq (T_\Sigma)^n$ is *automatic* if the tree language

$$\otimes R = \{ \otimes \bar{t} \mid \bar{t} \in R \} \subseteq T_{\Sigma_\square}$$

is regular. We say a tree automaton *recognises* R if it recognises $\otimes R$.

A (*relational*) *signature* $\tau = (\mathcal{R}, \text{ar})$ is a finite set \mathcal{R} of *relation symbols* together with an *arity map* $\text{ar}: \mathcal{R} \rightarrow \mathbb{N}_+$. A τ -structure $\mathfrak{A} = (A; (R^\mathfrak{A})_{R \in \mathcal{R}})$ consists of a set $A = \|\mathfrak{A}\|$, its *universe*, and an $\text{ar}(R)$ -ary relation $R^\mathfrak{A} \subseteq A^{\text{ar}(R)}$ for each $R \in \mathcal{R}$.² Given a subset $B \subseteq A$, the *induced substructure* $\mathfrak{A} \upharpoonright B$ is defined by

$$\|\mathfrak{A} \upharpoonright B\| = B \quad \text{and} \quad R^{\mathfrak{A} \upharpoonright B} = R^\mathfrak{A} \cap B^{\text{ar}(R)} \text{ for } R \in \mathcal{R}.$$

First order logic FO over τ is defined as usual and FO(\exists^∞) is its extension by the “there exist infinitely many”-quantifier \exists^∞ . Writing $\phi(x_1, \dots, x_n)$ means that all free variables of the formula ϕ are among the x_i . For a formula $\phi(x_1, \dots, x_m, y_1, \dots, y_n)$ and a tuple $\bar{b} \in A^n$ we let

$$\phi^\mathfrak{A}(\cdot, \bar{b}) = \{ \bar{a} \in A^m \mid \mathfrak{A} \models \phi(\bar{a}, \bar{b}) \}.$$

If $n = 0$ we simply write $\phi^\mathfrak{A}$ instead of $\phi^\mathfrak{A}(\cdot)$.

² By convention, structures are named in Fraktur and their universes by the same letter in Roman.

► **Definition 2.1.** A *tree-automatic presentation* of a τ -structure \mathfrak{A} is a tuple $(\mathcal{A}; (\mathcal{A}_R)_{R \in \mathcal{R}})$ of tree automata such that there exists a bijective *naming function* $\mu: A \rightarrow L(\mathcal{A})$ with the property that \mathcal{A}_R recognises $\mu(R^{\mathfrak{A}})$ for each $R \in \mathcal{R}$. A τ -structure is *tree-automatic* if it admits a tree-automatic presentation.

In the situation above, the structure $\mu(\mathfrak{A}) = (\mu(A); (\mu(R^{\mathfrak{A}}))_{R \in \mathcal{R}})$ is isomorphic to \mathfrak{A} and called a *tree-automatic copy* of \mathfrak{A} .

► **Theorem 2.2** (Blumensath [2]). *Let \mathfrak{A} be a tree-automatic structure, $\bar{\mathcal{A}}$ a tree-automatic presentation of \mathfrak{A} , μ the corresponding naming function, and $\phi(\bar{x})$ an $\text{FO}(\exists^\infty)$ -formula over τ . Then the relation $\mu(\phi^{\mathfrak{A}})$ is automatic and one can compute a tree automaton recognising it from $\bar{\mathcal{A}}$ and ϕ .*

► **Corollary 2.3** (Blumensath [2]). *Every tree-automatic structure possesses a decidable $\text{FO}(\exists^\infty)$ -theory.*

3 Delhommé's Decomposition Technique

In this section, we present the decomposition technique Delhommé used to show that every tree-automatic ordinal is below ω^{ω^ω} .

3.1 Sum and Box Augmentations and the Decomposition Theorem

The central notions of Delhommé's technique are sum augmentations and box augmentations.

► **Definition 3.1.** A τ -structure \mathfrak{A} is a *sum augmentation* of τ -structures $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ if there exists a finite partition $A = A_1 \uplus \dots \uplus A_n$ of \mathfrak{A} such that $\mathfrak{A}|_{A_i} \cong \mathfrak{B}_i$ for each $i = 1, \dots, n$.

► **Example 3.2.** Let $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ be linear orderings and \mathfrak{A} a linearisation of the partial ordering $\mathfrak{B}_1 \amalg \dots \amalg \mathfrak{B}_n = (\biguplus_{1 \leq i \leq n} B_i; \preceq)$ with $x \preceq y$ iff $x, y \in B_i$ and $x \leq^{\mathfrak{B}_i} y$ for some i . Then \mathfrak{A} is a sum augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$.

► **Remark.** Suppose a linear ordering $\mathfrak{A} = (A; \leq^{\mathfrak{A}})$ is a sum augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$. First, each \mathfrak{B}_i can be embedded into \mathfrak{A} and hence is a linear ordering itself. Moreover, if \mathfrak{A} is a well-ordering, then each \mathfrak{B}_i is a well-ordering too. Second, \mathfrak{A} is isomorphic to a linearisation of $\mathfrak{B}_1 \amalg \dots \amalg \mathfrak{B}_n$.

► **Definition 3.3.** A τ -structure \mathfrak{A} is a *box augmentation* of τ -structures $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ if there exists a bijection $f: B_1 \times \dots \times B_n \rightarrow A$ such that for all $j = 1, \dots, n$ and $\bar{x} \in \prod_{1 \leq i \leq n, i \neq j} B_i$ the map

$$f_{j, \bar{x}}: B_j \rightarrow A, b \mapsto (x_1, \dots, x_{j-1}, b, x_{j+1}, \dots, x_n)$$

is an embedding of \mathfrak{B}_j into \mathfrak{A} .

► **Example 3.4.** Let $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ be linear orderings and \mathfrak{A} a linearisation of the partial ordering $\mathfrak{B}_1 \times \dots \times \mathfrak{B}_n = (B_1 \times \dots \times B_n; \preceq)$ with $\bar{x} \preceq \bar{y}$ iff $x_i \leq^{\mathfrak{B}_i} y_i$ for all $i = 1, \dots, n$. Then \mathfrak{A} is a box augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$.

► **Remark.** Suppose a linear ordering \mathfrak{A} is a box augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$. First, each \mathfrak{B}_i can be embedded into \mathfrak{A} and hence is a linear ordering itself. Moreover, if \mathfrak{A} is a well-ordering, then each \mathfrak{B}_i is a well-ordering too. Second, the bijection f from Definition 3.3 above is an isomorphism between a linearisation of $\mathfrak{B}_1 \times \dots \times \mathfrak{B}_n$ and \mathfrak{A} .

Since the concept of box augmentations is too general for our purposes, we need to restrict it. In the following definition, an R -colouring of a τ -structure \mathfrak{B} is a map $c: B^{\text{ar}(R)} \rightarrow Q$ into a finite set Q such that $c(\bar{t}) \in c(R^{\mathfrak{B}})$ iff $\bar{t} \in R^{\mathfrak{B}}$ for all $\bar{t} \in B^{\text{ar}(R)}$.

► **Definition 3.5.** The box augmentation in Definition 3.3 is a *tame box augmentation* if for each $R \in \mathcal{R}$ the following condition holds: For every $i = 1, \dots, n$ there exists an R -colouring $c_i: B_i^{\text{ar}(R)} \rightarrow Q_i$ of \mathfrak{B}_i such that the map

$$\prod_{1 \leq i \leq n} Q_i, (f(\bar{x}_1), \dots, f(\bar{x}_r)) \mapsto (c_i(x_{1,i}, \dots, x_{r,i}))_{i=1, \dots, n}$$

is an R -colouring of \mathfrak{A} .

► **Remark.** Suppose a linear ordering \mathfrak{A} is a tame box augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$. For each $i = 1, \dots, n$ let $c_i: B_i^2 \rightarrow Q_i$ be the corresponding \leq -colouring of \mathfrak{B}_i . Without loss of generality, assume that the Q_i all are the same set, say $\{1, \dots, m\}$. For each $i = 1, \dots, n$ consider the structure $\mathfrak{C}_i = (B_i; R_1^{\mathfrak{C}_i}, \dots, R_m^{\mathfrak{C}_i})$ with $R_j^{\mathfrak{C}_i} = c_i^{-1}(j)$. Then the $R_j^{\mathfrak{C}_i}$ form a finite partition of B_i^2 which is compatible with $\leq^{\mathfrak{B}_i}$. Finally, the ordering \mathfrak{A} is a generalised product—in the sense of Feferman and Vaught—of the structures $\mathfrak{C}_1, \dots, \mathfrak{C}_n$ where only atomic formulae are used.

More generally, the very essence of the notion of a tame box augmentation is to first partition all relations as well as their complements and to take a generalised product afterwards.

► **Remark.** If \mathfrak{A} is a tame box augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ and $X_i \subseteq B_i$ for each i , then $\mathfrak{A}|f(X_1 \times \dots \times X_n)$ is tame box augmentation of $\mathfrak{B}_1|X_1, \dots, \mathfrak{B}_n|X_n$ via the bijection $f|(X_1 \times \dots \times X_n)$.

In the situations of Definitions 3.1, 3.3, and 3.5 we also say that the structures $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ form a *sum decomposition* respectively a *(tame) box decomposition* of \mathfrak{A} . The decomposition theorem for tree-automatic structures is the following, whose proof is postponed to Section 3.3.

► **Theorem 3.6** (Delhommé [4]). *Let \mathfrak{A} be a tree-automatic τ -structure and $\phi(x, y_1, \dots, y_n)$ an $\text{FO}(\exists^\infty)$ -formula over τ . Then there exists a finite set $\mathcal{S}_\phi^{\mathfrak{A}}$ of tree-automatic τ -structures such that for all $\bar{s} \in A^n$ the structure $\mathfrak{A}| \phi^{\mathfrak{A}}(\cdot, \bar{s})$ is a sum augmentation of tame box augmentations of elements from $\mathcal{S}_\phi^{\mathfrak{A}}$.*

For now, suppose that \mathcal{C} is a class of τ -structures ranked by ν , i.e., ν assigns to each structure $\mathfrak{A} \in \mathcal{C}$ an ordinal $\nu(\mathfrak{A})$, its ν -rank, which is invariant under isomorphism. An ordinal α is ν -sum-indecomposable if for any structure $\mathfrak{A} \in \mathcal{C}$ with $\nu(\mathfrak{A}) = \alpha$ every sum decomposition $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ of \mathfrak{A} contains a component \mathfrak{B}_i with $\mathfrak{B}_i \in \mathcal{C}$ and $\nu(\mathfrak{B}_i) = \alpha$. Similarly, we define ν -(tame)-box-indecomposable ordinals. Notice that every ν -box-indecomposable ordinal is also ν -tame-box-indecomposable. The following corollary is a direct consequence of Theorem 3.6.

► **Corollary 3.7** (Delhommé [4]). *Let \mathcal{C} be a class of τ -structures ranked by ν , \mathfrak{A} a tree-automatic τ -structure, and $\phi(x, y_1, \dots, y_n)$ an $\text{FO}(\exists^\infty)$ -formula over τ . Then there are only finitely many ordinals α which are simultaneously ν -sum-indecomposable as well as ν -tame-box-indecomposable and admit a $\bar{s} \in A^n$ with $\mathfrak{A}| \phi^{\mathfrak{A}}(\cdot, \bar{s}) \in \mathcal{C}$ and $\nu(\mathfrak{A}| \phi^{\mathfrak{A}}(\cdot, \bar{s})) = \alpha$.*

Proof. Let $\mathcal{S}_\phi^{\mathfrak{A}}$ be the finite set of structures which exists by Theorem 3.6. Consider an ordinal α which is ν -sum-indecomposable as well as ν -tame-box-indecomposable and admits a tuple $\bar{s} \in A^n$ with $\mathfrak{A}| \phi^{\mathfrak{A}}(\cdot, \bar{s}) \in \mathcal{C}$ and $\nu(\mathfrak{A}| \phi^{\mathfrak{A}}(\cdot, \bar{s})) = \alpha$. Then there exists a tame box decomposition $\mathfrak{B}_1, \dots, \mathfrak{B}_m$ of $\mathfrak{A}| \phi^{\mathfrak{A}}(\cdot, \bar{s})$ such that each \mathfrak{B}_i is a sum augmentation of

elements from $\mathcal{S}_\phi^\mathfrak{A}$. Since α is ν -tame-box-indecomposable, there is an $i_0 \in \{1, \dots, m\}$ such that $\mathfrak{B}_{i_0} \in \mathcal{C}$ and $\nu(\mathfrak{B}_{i_0}) = \alpha$. Moreover, there exists a sum decomposition $\mathfrak{C}_1, \dots, \mathfrak{C}_n$ of \mathfrak{B}_{i_0} such that $\mathfrak{C}_j \in \mathcal{S}_\phi^\mathfrak{A}$ for each $j = 1, \dots, n$. As α is also ν -sum-indecomposable, there is a $j_0 \in \{1, \dots, n\}$ such that $\mathfrak{C}_{j_0} \in \mathcal{C}$ and $\nu(\mathfrak{C}_{j_0}) = \alpha$.

In particular, $\mathcal{S}_\phi^\mathfrak{A}$ contains a structure \mathfrak{B} with $\mathfrak{B} \in \mathcal{C}$ and $\nu(\mathfrak{B}) = \alpha$. Since $\mathcal{S}_\phi^\mathfrak{A}$ is finite, there are only finitely many ordinals α of the type under consideration. \blacktriangleleft

3.2 Tree-Automatic Ordinals

In order to prove that every tree-automatic ordinal is strictly below ω^{ω^ω} , we apply Corollary 3.7 to the class of all well-orderings and rank each well-ordering \mathfrak{A} by its order type $\text{tp}(\mathfrak{A})$. To identify the tp-sum-indecomposable and tp-box-indecomposable ordinals, we need the natural sum and product.

Due to the Cantor normal form, every ordinal can be regarded as a polynomial in ω with natural numbers as coefficients and ordinals as exponents. Intuitively, the natural sum of two ordinals is formed by adding the corresponding polynomials and the natural product by multiplying the polynomials whereby exponents are added using the natural sum. Formally, let $\alpha = \sum_{i=1}^{i=n} \omega^{\gamma_i} k_i$ and $\beta = \sum_{i=1}^{i=n} \omega^{\gamma_i} \ell_i$ with $\gamma_1 > \dots > \gamma_n \geq 0$ and $k_1, \dots, k_n, \ell_1, \dots, \ell_n \in \mathbb{N}$ be two ordinals in Cantor normal form. The *natural sum* $\alpha \oplus \beta$ and the *natural product* $\alpha \otimes \beta$ are defined by

$$\alpha \oplus \beta = \sum_{i=1}^{i=n} \omega^{\gamma_i} (k_i + \ell_i) \quad \text{and} \quad \alpha \otimes \beta = \bigoplus_{i,j=1}^{i,j=n} \omega^{\gamma_i \oplus \gamma_j} k_i \ell_j.$$

Compared with the usual addition and multiplication of ordinals, both operations are commutative and strictly monotonic in both arguments and \otimes distributes over \oplus . The following theorem is an adaption of results in [3] to our setting.

► **Theorem 3.8** (Caruth [3]). *Let α and β_1, \dots, β_n be ordinals.*

1. *If α is a sum augmentation of β_1, \dots, β_n , then $\alpha \leq \beta_1 \oplus \dots \oplus \beta_n$.*
2. *If α is a box augmentation of β_1, \dots, β_n , then $\alpha \leq \beta_1 \otimes \dots \otimes \beta_n$.*

► **Corollary 3.9.** *Let α be an ordinal. Then ω^α is tp-sum-indecomposable and ω^{ω^α} is tp-box-indecomposable.*

Proof. Let β_1, \dots, β_n be a sum decomposition of ω^α . Then $\beta_i \leq \omega^\alpha$ for each i . If $\beta_i < \omega^\alpha$ for all i , then $\beta_1 \oplus \dots \oplus \beta_n < \omega^\alpha$. This contradicts Theorem 3.8 (1).

Now, let β_1, \dots, β_n be a box decomposition of ω^{ω^α} . Then $\beta_i \leq \omega^{\omega^\alpha}$ for each i . By contradiction, assume $\beta_i < \omega^{\omega^\alpha}$ for all i . Since ω^{ω^α} is a limit ordinal, there are $\gamma_i < \omega^\alpha$ with $\beta_i < \omega^{\gamma_i}$ and hence

$$\beta_1 \otimes \dots \otimes \beta_n < \omega^{\gamma_1 \oplus \dots \oplus \gamma_n} < \omega^{\omega^\alpha}.$$

This contradicts Theorem 3.8 (2). \blacktriangleleft

Finally, Corollaries 3.7 and 3.9 imply that any tree-automatic ordinal is strictly less than ω^{ω^ω} . The main ingredient for the converse implication is the following lemma.

► **Lemma 3.10.** *For each $k \in \mathbb{N}$ the ordinal ω^{ω^k} admits a tree-automatic presentation over a unary alphabet Σ .*

Proof. We proceed by induction on k .

Base case. $k = 0$.

The map $\mu: \omega \rightarrow T_\Sigma$ which assigns to $n \in \omega$ the unique tree $\mu(n)$ with $\text{dom}(\mu(n)) = \{0\}^{<n}$ can be used as naming function for a tree-automatic presentation of ω .

Inductive step. $k > 0$.

We regard ω^{ω^k} as the length-lexicographically ordered set of all maps $f: \omega \rightarrow \omega^{\omega^{k-1}}$ which are zero almost everywhere. Let ν be the naming function corresponding to the tree-automatic presentation of $\omega^{\omega^{k-1}}$ which exists by induction. We define a map $\mu: \omega^{\omega^k} \rightarrow T_\Sigma$ by letting $\mu(f)$ be the unique tree with

$$\text{dom}(\mu(f)) = \bigcup_{0 \leq i < n} \{0^i\} \cup \{0^i 1\} \text{dom}(\nu(f(i))),$$

where $n \in \omega$ is minimal with $f(m) = 0$ for all $m \geq n$. This map can be used as naming function for a tree-automatic presentation of ω^{ω^k} . \blacktriangleleft

► **Corollary 3.11** (Delhommé [4]). *An ordinal α is tree-automatic if, and only if,*

$$\alpha < \omega^{\omega^\omega}.$$

Proof. By contradiction, assume there exists a tree-automatic ordinal $\alpha \geq \omega^{\omega^\omega}$. Consider $\phi(x, y) = x \leq y \wedge x \neq y$. Clearly, $\phi^\alpha(\cdot, \beta) = \beta$ for every $\beta \in \alpha$. In particular, $\text{tp}(\alpha \upharpoonright \phi^\alpha(\cdot, \omega^{\omega^d})) = \omega^{\omega^d}$ for each $d \in \mathbb{N}$. Since these ordinals ω^{ω^d} are tp-sum-indecomposable as well as tp-box-indecomposable, this contradicts Corollary 3.7.

Now, let $\alpha < \omega^{\omega^\omega}$ be some ordinal. There exists a $k \in \mathbb{N}$ such that $\alpha < \omega^{\omega^k}$. By Lemma 3.10, ω^{ω^k} is tree-automatic. Finally, α is FO-definable with one parameter in ω^{ω^k} and hence tree-automatic. \blacktriangleleft

3.3 Proof of the Decomposition Theorem

We conclude this section by providing a proof of Theorem 3.6.

Proof of Theorem 3.6. Let $(\mathcal{A}; (\mathcal{A}_R)_{R \in \mathcal{R}})$ be a tree-automatic presentation of \mathfrak{A} with $L(\mathcal{A}) \subseteq T_\Sigma$. To keep notation simple, we assume that the corresponding naming function $\mu: A \rightarrow L(\mathcal{A})$ is the identity, i.e., \mathfrak{A} is identified with its tree-automatic copy $\mu(\mathfrak{A})$. For $R \in \mathcal{R}$ let Q_R be the set of states of \mathcal{A}_R . Moreover, let \mathcal{A}_ϕ be a tree automaton recognising $\phi^\mathfrak{A}$ and Q_ϕ its set of states. For each $t \in T_\Sigma$ and all $r \geq 1$ we put $\otimes_r t = \otimes(t, \dots, t) \in T_{\Sigma_\square^r}$, where the convolution is made up of r copies of t . We further define a tree $\boxtimes_n t = (t, \emptyset, \dots, \emptyset) \in T_{\Sigma_\square^{1+n}}$, where the number of empty trees \emptyset in the convolution is n . To simplify notation even more, we put

$$\llbracket t \rrbracket_\phi = \mathcal{A}_\phi(\boxtimes_n t) \quad \text{and} \quad \llbracket t \rrbracket_R = \mathcal{A}_R(\otimes_{\text{ar}(R)} t)$$

for every $t \in T_\Sigma$ and $R \in \mathcal{R}$.

Consider the set

$$\Gamma = \prod_{R \in \{\phi\} \uplus \mathcal{R}} Q_R \times \prod_{R \in \mathcal{R}} 2^{Q_R}.$$

For each $\gamma = ((q_R)_{R \in \{\phi\} \uplus \mathcal{R}}, (P_R)_{R \in \mathcal{R}}) \in \Gamma$ we define a structure \mathfrak{S}_γ by

$$\|\mathfrak{S}_\gamma\| = S_\gamma = \{ t \in T_\Sigma \mid \llbracket t \rrbracket_\phi = q_\phi \text{ and } \llbracket t \rrbracket_R = q_R \text{ for each } R \in \mathcal{R} \}$$

and

$$R^{\mathfrak{S}_\gamma} = \left\{ \bar{t} \in S_\gamma^{\text{ar}(R)} \mid \mathcal{A}_R(\otimes \bar{t}) \in P_R \right\} \text{ for } R \in \mathcal{R}.$$

Clearly, \mathfrak{S}_γ is a tree-automatic copy of itself. Finally, we put

$$\mathcal{S}_\phi^{\mathfrak{A}} = \{ \mathfrak{S}_\gamma \mid \gamma \in \Gamma \}.$$

Obviously, this set is finite.

For the rest of this proof, we fix some parameters $\bar{s} = (s_1, \dots, s_n) \in A^n$ and put $D = \bigcup_{1 \leq i \leq n} \text{dom}(s_i)$. The \bar{s} -type of a tree $t \in T_\Sigma$ is the tuple

$$\text{tp}_{\bar{s}}(t) = (t \upharpoonright D, U, (\rho_R)_{R \in \{\phi\} \uplus \mathcal{R}}),$$

where $t \upharpoonright D \in T_\Sigma$ is the restriction of t to the tree domain $\text{dom}(t) \cap D$, $U = \text{dom}(t) \cap \partial D$, and $\rho_R: U \rightarrow Q_R, u \mapsto \llbracket t \upharpoonright u \rrbracket_R$ for each $R \in \{\phi\} \uplus \mathcal{R}$. Observe that

$$\otimes(t, \bar{s}) = \otimes(t \upharpoonright D, \bar{s})[(u / \boxtimes_n t \upharpoonright u)_{u \in U}]$$

and hence

$$\mathcal{A}_\phi(\otimes(t, \bar{s})) = \mathcal{A}_\phi(\otimes(t \upharpoonright D, \bar{s}), \rho_\phi), \quad (1)$$

i.e., whether $t \in \phi^{\mathfrak{A}}(\cdot, \bar{s})$ is valid can be determined from $\text{tp}_{\bar{s}}(t)$. Since D is finite, there are only finitely many distinct \bar{s} -types. Consequently, the equivalence relation $\sim_{\bar{s}}$ on T_Σ defined by $t \sim_{\bar{s}} t'$ iff $\text{tp}_{\bar{s}}(t) = \text{tp}_{\bar{s}}(t')$ has finite index. Due to Eq. (1), $\phi^{\mathfrak{A}}(\cdot, \bar{s})$ is a union of $\sim_{\bar{s}}$ -classes. Say $B_1, \dots, B_m \subseteq \phi^{\mathfrak{A}}(\cdot, \bar{s})$ are these $\sim_{\bar{s}}$ -classes, then $\mathfrak{A} \upharpoonright \phi^{\mathfrak{A}}(\cdot, \bar{s})$ is a sum augmentation of $\mathfrak{A} \upharpoonright B_1, \dots, \mathfrak{A} \upharpoonright B_m$. Thus, it remains to show that $\mathfrak{A} \upharpoonright B$ is a tame box augmentation of elements from $\mathcal{S}_\phi^{\mathfrak{A}}$ for each $\sim_{\bar{s}}$ -class $B \subseteq \phi^{\mathfrak{A}}(\cdot, \bar{s})$.

Therefore, fix some $\sim_{\bar{s}}$ -class $B \subseteq \phi^{\mathfrak{A}}(\cdot, \bar{s})$, let $\vartheta = (t_D, U, (\rho_R)_{R \in \{\phi\} \uplus \mathcal{R}})$ be the corresponding \bar{s} -type, and put $\mathfrak{B} = \mathfrak{A} \upharpoonright B$. For $u \in U$ we define

$$\gamma(\vartheta, u) = ((\rho_R(u))_{R \in \{\phi\} \uplus \mathcal{R}}, (P_R(u))_{R \in \mathcal{R}}) \in \Gamma$$

by

$$P_R(u) = \{ q \in Q_R \mid \mathcal{A}_R(\otimes_{\text{ar}(R)} t_D, \rho_R[u \mapsto q]) \in F_R \} \text{ for } R \in \mathcal{R},$$

where $F_R \subseteq Q_R$ is the set of accepting states of \mathcal{A}_R . Let u_1, \dots, u_m be an enumeration of the elements of U and put $\mathfrak{C}_i = \mathfrak{S}_{\gamma(\vartheta, u_i)}$ for $i = 1, \dots, m$. Next, we show that \mathfrak{B} is a tame box augmentation of $\mathfrak{C}_1, \dots, \mathfrak{C}_m$.

First, observe that

$$f: C_1 \times \dots \times C_m \rightarrow T_\Sigma, (x_1, \dots, x_m) \mapsto t_D[u_1/x_1, \dots, u_m/x_m]$$

is injective. Some $t \in T_\Sigma$ is contained in the image of f if, and only if, $t \upharpoonright D = t_D$, $\text{dom}(t) \cap \partial D = U$, and $t \upharpoonright u_i \in C_i$ for each $i = 1, \dots, m$. The latter is equivalent to $\text{tp}_{\bar{s}}(t) = \vartheta$ and hence f is a bijection $f: C_1 \times \dots \times C_m \rightarrow B$. Fix some $j = 1, \dots, m$ and $\bar{x} \in \prod_{1 \leq i \leq m, i \neq j} C_i$ and let

$$f_{j, \bar{x}}: C_j \rightarrow B, t \mapsto f(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_m).$$

Consider $R \in \mathcal{R}$ and $r = \text{ar}(R)$. For all $\bar{t} \in C_j^r$ we have

$$\otimes f_{j,\bar{x}}(\bar{t}) = (\otimes_r t_D) [(u_i / \otimes_r x_i)_{1 \leq i \leq m, i \neq j}, u_j / \otimes \bar{t}]$$

and hence

$$\mathcal{A}_R(\otimes f_{j,\bar{x}}(\bar{t})) = \mathcal{A}_R(\otimes_r t_D, \rho_R[u_j \mapsto \mathcal{A}_R(\otimes \bar{t})]).$$

This leads to the following chain of equivalences

$$\begin{aligned} f_{j,\bar{x}}(\bar{t}) \in R^{\mathfrak{B}} &\iff \mathcal{A}_R(\otimes f_{j,\bar{x}}(\bar{t})) \in F_R \\ &\iff \mathcal{A}_R(\otimes_r t_D, \rho_R[u_j \mapsto \mathcal{A}_R(\otimes \bar{t})]) \in F_R \\ &\iff \mathcal{A}_R(\otimes \bar{t}) \in P_R(u_j) \\ &\iff \bar{t} \in R^{\mathfrak{C}_j}, \end{aligned}$$

which shows that \mathfrak{B} is a box augmentation of $\mathfrak{C}_1, \dots, \mathfrak{C}_m$. It remains to show that this box augmentation is tame.

Therefore, fix some $R \in \mathcal{R}$, put $r = \text{ar}(R)$, and notice that the map

$$c_i: C_i^r \rightarrow Q_R, \bar{t} \mapsto \mathcal{A}_R(\otimes \bar{t})$$

is an R -colouring of \mathfrak{C}_i for each $i = 1, \dots, m$. We have to show that

$$c: B^r \rightarrow Q_R^m, (f(\bar{x}_1), \dots, f(\bar{x}_r)) \mapsto (c_i(x_{1,i}, \dots, x_{r,i}))_{1 \leq i \leq m}$$

is an R -colouring of \mathfrak{B} . Consider the map

$$h: Q_R^m \rightarrow Q_m, (q_1, \dots, q_m) \mapsto \mathcal{A}_R(\otimes_r t_D, \{u_i \mapsto q_i \mid 1 \leq i \leq m\}).$$

For every $\bar{t} \in B^r$ we obtain $h(c(\bar{t})) = \mathcal{A}_R(\otimes \bar{t})$ and hence $h \circ c$ is an R -colouring of \mathfrak{B} . Consequently, c is an R -colouring of \mathfrak{B} as well. \blacktriangleleft

4 Tree-Automatic Linear Orderings

The objective of this section is to prove our main result, namely Theorem 4.4, which states that every tree-automatic linear ordering has FC-rank below ω^ω . Due to the fact that every countable linear ordering is a dense sum of scattered linear orderings, the proof is essentially an application of Corollary 3.7 to the class of countable scattered linear orderings ranked by VD_* , a variation of the FC-rank. Since it is already known that every ordinal is VD_* -sum-indecomposable [7], the major part of this section is devoted to identifying the VD_* -tame-box-indecomposable ordinals.

4.1 Linear Orderings and the FC-rank

A (linear) ordering is a structure $\mathfrak{A} = (A; \leq^{\mathfrak{A}})$ where $\leq^{\mathfrak{A}}$ is a *non-strict* linear order on A . Sometimes we use the corresponding *strict* linear order $<^{\mathfrak{A}}$. If \mathfrak{A} is clear from the context we omit the superscript \mathfrak{A} . An *interval* in \mathfrak{A} is a subset $I \subseteq A$ such that $x < z < y$ implies $z \in I$ for all $x, y \in I$ and $z \in A$. For $x, y \in A$ the *closed interval* $[x, y]_{\mathfrak{A}}$ in \mathfrak{A} is the set $\{z \in A \mid x \leq z \leq y\}$ if $x \leq y$ and the set $\{z \in A \mid y \leq z \leq x\}$ if $x > y$.

► **Definition 4.1.** A *condensation (relation)* on a linear ordering \mathfrak{A} is an equivalence relation \sim on A such that each \sim -class is an interval of \mathfrak{A} .

For two subsets $X, Y \subseteq A$ we write $X \ll Y$ if $x < y$ for all $x \in X$ and $y \in Y$. If \sim is a condensation on \mathfrak{A} , the set A/\sim of all \sim -classes is (strictly) linearly ordered by \ll . We denote the corresponding linear ordering by \mathfrak{A}/\sim . An example of a condensation is the relation \sim with $x \sim y$ iff the closed interval $[x, y]_{\mathfrak{A}}$ in \mathfrak{A} is finite. The ordering \mathfrak{A}/\sim is obtained from \mathfrak{A} by identifying points which are only finitely far away from each other. If this process is transfinitely iterated, it eventually becomes stationary. Intuitively, the FC-rank of \mathfrak{A} is the ordinal α counting the number of steps which are necessary to reach this fix point.

► **Definition 4.2.** Let \mathfrak{A} be a linear ordering. For each ordinal α a condensation $\sim_{\alpha}^{\mathfrak{A}}$ on \mathfrak{A} is defined by transfinite induction:

1. $\sim_0^{\mathfrak{A}}$ is the identity relation on \mathfrak{A} ,
2. for successor ordinals $\alpha = \beta + 1$ let $x \sim_{\alpha}^{\mathfrak{A}} y$ iff the interval $[\tilde{x}, \tilde{y}]_{\mathfrak{A}/\sim_{\beta}^{\mathfrak{A}}}$ in $\mathfrak{A}/\sim_{\beta}^{\mathfrak{A}}$ is finite, where \tilde{x} and \tilde{y} are the $\sim_{\beta}^{\mathfrak{A}}$ -classes of x and y , and
3. for limit ordinals α let $x \sim_{\alpha}^{\mathfrak{A}} y$ iff $x \sim_{\beta}^{\mathfrak{A}} y$ for some $\beta < \alpha$.

For each ordering \mathfrak{A} there exists an ordinal α such that $\sim_{\alpha}^{\mathfrak{A}}$ and $\sim_{\beta}^{\mathfrak{A}}$ coincide for each $\beta \geq \alpha$. More precisely, every ordinal α whose cardinality is greater than the one of \mathfrak{A} has this property. Theorem 5.9 in [8] ascertains that if \mathfrak{A} is countable then α can be chosen countable as well.

► **Definition 4.3.** The FC-rank of a linear ordering \mathfrak{A} , denoted by $\text{FC}(\mathfrak{A})$, is the least ordinal α such that $\sim_{\alpha}^{\mathfrak{A}}$ and $\sim_{\beta}^{\mathfrak{A}}$ coincide for each $\beta \geq \alpha$.

For a linear ordering \mathfrak{A} and a subset $B \subseteq A$ we simply write $\text{FC}(B)$ for $\text{FC}(\mathfrak{A}|B)$. The following theorem is the main result of this article.

► **Theorem 4.4.** Let \mathfrak{A} be a tree-automatic linear ordering. Then

$$\text{FC}(\mathfrak{A}) < \omega^{\omega}.$$

Since $\text{FC}(\alpha) \leq \beta$ if, and only if, $\alpha \leq \omega^{\beta}$ for all countable ordinals α and β , Theorem 4.4 above yields another proof of the fact that every tree-automatic ordinal is strictly less than $\omega^{\omega^{\omega}}$ (cf. Corollary 3.11).

4.2 Scattered Linear Orderings and the VD-rank

Throughout the rest of this paper, we consider only countable linear orderings. A linear ordering \mathfrak{A} is *scattered* if the ordering $(\mathbb{Q}; <)$ of the rationals cannot be embedded into \mathfrak{A} , or equivalently, if there exists an ordinal α such that $\mathfrak{A}/\sim_{\alpha}^{\mathfrak{A}}$ contains exactly one element (cf. Chapter 5 in [8]). Examples of scattered orderings include the natural numbers $\omega = (\mathbb{N}; \leq)$, the reversed natural numbers $\omega^* = (\mathbb{N}; \geq)$, the integers $\zeta = (\mathbb{Z}; \leq)$, and the finite linear orderings $\mathbf{n} = (\{1, \dots, n\}; \leq)$ for $n \in \mathbb{N}$. Furthermore, every ordinal is scattered.

For an ordering \mathfrak{J} the \mathfrak{J} -sum of an I -indexed family $(\mathfrak{A}_i)_{i \in I}$ of orderings is the linear ordering

$$\mathfrak{A} = \sum_{i \in \mathfrak{J}} \mathfrak{A}_i$$

defined by $A = \biguplus_{i \in I} A_i$ and $x \leq^{\mathfrak{A}} y$ iff $x, y \in A_i$ and $x \leq^{\mathfrak{A}_i} y$ for some $i \in I$ or $x \in A_i$ and $y \in A_j$ for some $i, j \in I$ with $i <^{\mathfrak{J}} j$. If \mathfrak{J} is finite, say $\mathfrak{J} = \mathbf{n}$, we write $\mathfrak{A}_1 + \dots + \mathfrak{A}_n$ for $\sum_{i \in \mathbf{n}} \mathfrak{A}_i$.

Next, we introduce the class of very discrete linear orderings and their connection to the scattered linear orderings.

► **Definition 4.5.** For each countable ordinal α the class \mathcal{VD}_α of linear orderings is defined by transfinite induction:

1. $\mathcal{VD}_0 = \{\mathbf{0}, \mathbf{1}\}$, and
2. for $\alpha > 0$ the class \mathcal{VD}_α contains all finite sums, ω -sums, ω^* -sums, and ζ -sums of elements from $\mathcal{VD}_{<\alpha} = \bigcup_{\beta < \alpha} \mathcal{VD}_\beta$.

The class \mathcal{VD} of *very discrete* linear orderings is the union of all classes \mathcal{VD}_α . The *VD-rank* of some $\mathfrak{A} \in \mathcal{VD}$, denoted by $\text{VD}(\mathfrak{A})$, is the least ordinal α with $\mathfrak{A} \in \mathcal{VD}_\alpha$.

The following result is due to Hausdorff and Theorem 5.24 in [8].

► **Theorem 4.6** (Hausdorff [5]). *A countable linear ordering \mathfrak{A} is scattered if, and only if, it is contained in \mathcal{VD} . In case \mathfrak{A} is scattered,*

$$\text{FC}(\mathfrak{A}) = \text{VD}(\mathfrak{A}).$$

In order to formulate the intermediate steps of our proof of Theorem 4.4, we need a slight variation of the VD-rank [7].

► **Definition 4.7.** The *VD_{*}-rank* of a scattered linear ordering \mathfrak{A} , denoted by $\text{VD}_*(\mathfrak{A})$, is the least ordinal α such that \mathfrak{A} is a finite sum of elements from \mathcal{VD}_α .

The VD-rank and the VD_{*}-rank of a scattered linear ordering \mathfrak{A} are closely related by the following inequality

$$\text{VD}_*(\mathfrak{A}) \leq \text{VD}(\mathfrak{A}) \leq \text{VD}_*(\mathfrak{A}) + 1. \quad (2)$$

The following lemma is very useful when reasoning about the ranks of scattered linear orderings. The first inequality is Lemma 5.14 in [8] and the second inequality is a trivial consequence of the first one.

► **Lemma 4.8.** *Let \mathfrak{A} be a scattered linear ordering and $B \subseteq A$. Then*

$$\text{VD}(\mathfrak{A} \upharpoonright B) \leq \text{VD}(\mathfrak{A}) \quad \text{and} \quad \text{VD}_*(\mathfrak{A} \upharpoonright B) \leq \text{VD}_*(\mathfrak{A}).$$

4.3 Sum and Box Augmentations of Scattered Linear Orderings

Every sum decomposition of a scattered linear ordering \mathfrak{A} entirely consists of scattered linear orderings (cf. Remark 3.1). The relationship between the VD_{*}-ranks of \mathfrak{A} and the components was established in [7].

► **Proposition 4.9** (Khoussainov, Rubin, Stephan [7]). *Let \mathfrak{A} be a scattered linear ordering and a sum augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$. Then*

$$\text{VD}_*(\mathfrak{A}) = \max\{\text{VD}_*(\mathfrak{B}_1), \dots, \text{VD}_*(\mathfrak{B}_n)\}.$$

► **Corollary 4.10.** *Every countable ordinal is VD_{*}-sum-indecomposable.*

As already mentioned, we are mainly interested in the VD_{*}-tame-box-indecomposable ordinals. The main tool for identifying them is Proposition 4.11 below whose proof is postponed to page 15. Notice that Remark 3.1 implies that $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ therein are scattered linear orderings.

► **Proposition 4.11.** *Let \mathfrak{A} be a scattered linear ordering and a tame box augmentation of $\mathfrak{B}_1, \dots, \mathfrak{B}_n$. Then*

$$\text{VD}_*(\mathfrak{A}) \leq \text{VD}_*(\mathfrak{B}_1) \oplus \dots \oplus \text{VD}_*(\mathfrak{B}_n).$$

► **Corollary 4.12.** *Every countable ordinal of the shape ω^α is VD_* -tame-box-indecomposable.*

Proof. Let \mathfrak{A} be a scattered linear ordering with $\text{VD}_*(\mathfrak{A}) = \omega^\alpha$ and $\mathfrak{B}_1, \dots, \mathfrak{B}_n$ a tame box decomposition of \mathfrak{A} . Since each \mathfrak{B}_i can be embedded into \mathfrak{A} , Lemma 4.8 yields $\text{VD}_*(\mathfrak{B}_i) \leq \omega^\alpha$. If $\text{VD}_*(\mathfrak{B}_i) < \omega^\alpha$ for each i , then

$$\text{VD}_*(\mathfrak{B}_1) \oplus \dots \oplus \text{VD}_*(\mathfrak{B}_n) < \omega^\alpha.$$

This contradicts Proposition 4.11. ◀

As a first step towards the proof of Proposition 4.11 we provide two rather technical lemmas.

► **Lemma 4.13.** *Let \mathfrak{A} be a linear ordering without a greatest element and $c: A^2 \rightarrow Q$ a \leq -colouring of \mathfrak{A} . Then there exist a strictly increasing, unbounded sequence $(a_i)_{i \in \mathbb{N}}$ in \mathfrak{A} and a colour $q \in Q$ such that $c(a_i, a_j) = q$ for all $i, j \in \mathbb{N}$ with $i < j$.*

Proof. Since \mathfrak{A} has no greatest element, there exists a strictly increasing and unbounded sequence $(x_i)_{i \in \mathbb{N}}$ in \mathfrak{A} . By Ramsey's theorem for infinite, undirected, edge coloured graphs there exist an infinite set $H \subseteq \mathbb{N}$ and a colour $q \in Q$ such that $c(x_i, x_j) = q$ for all $i, j \in H$ with $i < j$. Let $k_0 < k_1 < \dots$ be the increasing enumeration of all elements in H and put $a_i = x_{k_i}$ for all $i \in \mathbb{N}$. ◀

Notice that the dual of this lemma holds as well and makes a statement about linear orderings without a least element and strictly decreasing, unbounded sequences. In the following lemma, the interval $(-\infty, a_0]_{\mathfrak{A}}$ denotes the set of all $a \in A$ with $a \leq a_0$.

► **Lemma 4.14.** *Let \mathfrak{A} be an ω -sum of elements from $\mathcal{VD}_{<\alpha}$ and $(a_i)_{i \in \mathbb{N}}$ a increasing sequence in \mathfrak{A} . Then*

$$\text{VD}_*((-\infty, a_0]_{\mathfrak{A}}) < \alpha \quad \text{and} \quad \text{VD}_*((a_{k-1}, a_k]_{\mathfrak{A}}) < \alpha \quad \text{for all } k \geq 1.$$

Proof. Let $\mathfrak{A} = \sum_{i \in \omega} \mathfrak{A}_i$ with $\mathfrak{A}_i \in \mathcal{VD}_{<\alpha}$ for all $i \in \omega$. For each $k \in \omega$ there exists a unique $\ell \in \omega$ with $a_k \in A_\ell$. Then $(-\infty, a_k]_{\mathfrak{A}} \subseteq A_0 \cup \dots \cup A_\ell$ and hence

$$\text{VD}_*((-\infty, a_k]_{\mathfrak{A}}) \leq \text{VD}_*(\mathfrak{A}_0 + \dots + \mathfrak{A}_\ell) < \alpha.$$

Moreover, for $k \geq 1$ we have $\text{VD}_*((a_{k-1}, a_k]_{\mathfrak{A}}) \leq \text{VD}_*((-\infty, a_k]_{\mathfrak{A}}) < \alpha$. ◀

Again, the dual of this statement which speaks about ω^* -sums and decreasing sequences holds true. Basically, the proof of Proposition 4.11 proceeds by induction on n and reduces thus to the case $n = 2$. Proposition 4.15 slightly rephrases the claim for $n = 2$.

► **Proposition 4.15.** *Let α and β be ordinals, \mathfrak{C} a scattered linear ordering, and \mathfrak{A} and \mathfrak{B} form a tame box decomposition of \mathfrak{C} with $\text{VD}_*(\mathfrak{A}) \leq \alpha$ and $\text{VD}_*(\mathfrak{B}) \leq \beta$. Then*

$$\text{VD}_*(\mathfrak{C}) \leq \alpha \oplus \beta. \tag{3}$$

Proof. We proceed by induction on α and β . To keep notation simple, we assume that the map $f: A \times B \rightarrow C$ from the definition of box augmentation is the identity, i.e., $C = A \times B$ and \mathfrak{C} is a linearisation of $\mathfrak{A} \times \mathfrak{B}$ (cf. Remark 3.1).

Before delving into the induction, we perform a slight simplification. By definition, there exist $\mathfrak{A}_1, \dots, \mathfrak{A}_m \in \mathcal{VD}_\alpha$ and $\mathfrak{B}_1, \dots, \mathfrak{B}_n \in \mathcal{VD}_\beta$ such that $\mathfrak{A} = \mathfrak{A}_1 + \dots + \mathfrak{A}_m$ and $\mathfrak{B} = \mathfrak{B}_1 + \dots + \mathfrak{B}_n$. Since every ζ -sum of linear orderings can be written as a sum of an ω -sum and an ω^* -sum, we can assume that none of the \mathfrak{A}_i or \mathfrak{B}_j is constructed as a ζ -sum.

Obviously, \mathfrak{C} is a sum augmentation of the $m \cdot n$ orderings $\mathfrak{C} \upharpoonright (A_i \times B_j)$. By Proposition 4.9, it suffices to show

$$\text{VD}_*(\mathfrak{C} \upharpoonright (A_i \times B_j)) \leq \alpha \oplus \beta$$

for all i and j . Since $\mathfrak{C} \upharpoonright (A_i \times B_j)$ is a tame box augmentation of \mathfrak{A}_i and \mathfrak{B}_j , it remains to show Eq. (3) under the stronger assumptions that $\text{VD}(\mathfrak{A}) \leq \alpha$, $\text{VD}(\mathfrak{B}) \leq \beta$, and neither \mathfrak{A} nor \mathfrak{B} is constructed as a ζ -sum.

Base case. $\alpha = 0$ or $\beta = 0$.

If $\alpha = 0$, then $\mathfrak{A} \cong \mathbf{1}$ and $\mathfrak{C} \cong \mathfrak{B}$. Thus, $\text{VD}_*(\mathfrak{C}) = \text{VD}_*(\mathfrak{B}) \leq \alpha \oplus \beta$. Similarly, $\text{VD}_*(\mathfrak{C}) \leq \alpha \oplus \beta$ if $\beta = 0$.

Inductive step. $\alpha > 0$ and $\beta > 0$.

If \mathfrak{A} is a finite sum of elements from $\mathcal{VD}_{<\alpha}$, then $\text{VD}_*(\mathfrak{A}) < \alpha$ and $\text{VD}_*(\mathfrak{C}) < \alpha \oplus \beta$ by induction. Similarly, $\text{VD}_*(\mathfrak{C}) < \alpha \oplus \beta$ if \mathfrak{B} is a finite sum. It remains to show the claim under the assumption that \mathfrak{A} and \mathfrak{B} are ω -sums or ω^* -sums. We distinguish four cases. In each case, let $c_1: A^2 \rightarrow Q_1$ and $c_2: B^2 \rightarrow Q_2$ be \leq -colourings of \mathfrak{A} and \mathfrak{B} such that

$$c: (A \times B)^2 \rightarrow Q_1 \times Q_2, ((a_1, b_1), (a_2, b_2)) \mapsto (c_1(a_1, a_2), c_2(b_1, b_2))$$

is a \leq -colouring of \mathfrak{C} .

Case 1. \mathfrak{A} is an ω -sum of elements from $\mathcal{VD}_{<\alpha}$ and \mathfrak{B} is an ω^* -sum of elements from $\mathcal{VD}_{<\beta}$. By Lemma 4.13, there exist a strictly increasing, unbounded sequence $(a_i)_{i \in \mathbb{N}}$ in \mathfrak{A} and a colour $q_1 \in Q_1$ such that $c_1(a_i, a_j) = q_1$ for all $i, j \in \mathbb{N}$ with $i < j$. By the dual of Lemma 4.13, there exist a strictly decreasing, unbounded sequence $(b_i)_{i \in \mathbb{N}}$ in \mathfrak{B} and a colour $q_2 \in Q_2$ such that $c_2(b_i, b_j) = q_2$ for all $i, j \in \mathbb{N}$ with $i > j$. Depending on how (a_0, b_0) compares to (a_1, b_1) in \mathfrak{C} , we distinguish two cases.

Case 1.1. $(a_0, b_0) < (a_1, b_1)$.

Figure 1 depicts the idea behind the treatment of this case. The horizontal axis describes \mathfrak{A} and increases from left to right, whereas the vertical axis outlines \mathfrak{B} and grows from bottom to top. Within the grid, arrows point from smaller to greater elements.

Formally, let

$$X_0 = (-\infty, a_0]_{\mathfrak{A}} \times (-\infty, b_0)_{\mathfrak{B}} \quad X_k = (a_{k-1}, a_k]_{\mathfrak{A}} \times (-\infty, b_0)_{\mathfrak{B}} \text{ for } k \geq 1$$

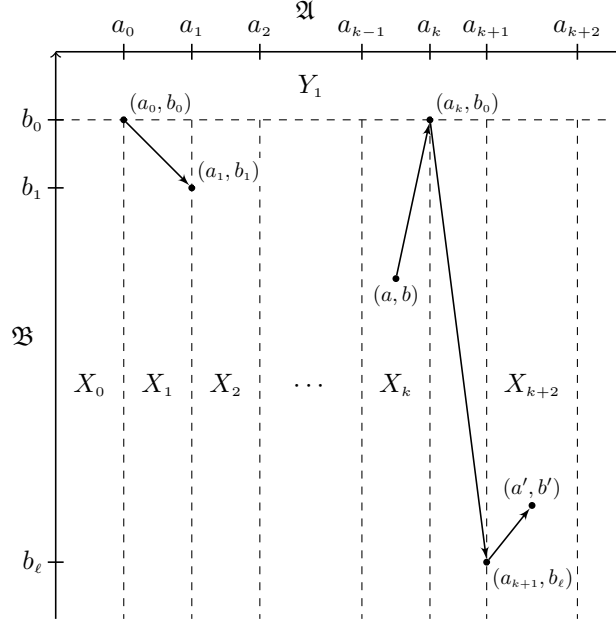
and

$$Y_1 = A \times [b_0, \infty)_{\mathfrak{B}} \quad Y_2 = \bigcup_{k \in \mathbb{N}} X_{2k} \quad Y_3 = \bigcup_{k \in \mathbb{N}} X_{2k+1}.$$

Since $A \times B = Y_1 \uplus Y_2 \uplus Y_3$, by Proposition 4.9, it suffices to show $\text{VD}_*(Y_i) \leq \alpha \oplus \beta$ for $i = 1, 2, 3$. Lemma 4.14 and its dual yield

$$\text{VD}_*((-\infty, a_0]_{\mathfrak{A}}) < \alpha \quad \text{VD}_*((a_{k-1}, a_k]_{\mathfrak{A}}) < \alpha \text{ for } k \geq 1 \quad \text{VD}_*([b_0, \infty)_{\mathfrak{B}}) < \beta.$$

Together with the induction hypothesis this yields $\text{VD}_*(X_k) < \alpha \oplus \beta$ for all $k \in \mathbb{N}$ as well as $\text{VD}_*(Y_1) < \alpha \oplus \beta$.



■ **Figure 1** Proof sketch for Case 1.1.

As a next step, we show that

$$X_k \ll X_{k+2} \text{ for all } k \in \mathbb{N}. \quad (4)$$

Therefore, let $(a, b) \in X_k$ and $(a', b') \in X_{k+2}$. Since the sequence of the b_i is strictly decreasing and unbounded, there is an $\ell \geq 1$ such that $b_\ell \leq b'$. The choice of the sequences $(a_i)_{i \in \mathbb{N}}$ and $(b_i)_{i \in \mathbb{N}}$ implies

$$c((a_0, b_0), (a_1, b_1)) = (q_1, q_2) = c((a_k, b_0), (a_{k+1}, b_\ell))$$

and hence $(a_k, b_0) < (a_{k+1}, b_\ell)$. Since \mathfrak{C} is a linearisation of $\mathfrak{A} \times \mathfrak{B}$, we have $(a, b) < (a_k, b_0)$ and $(a_{k+1}, b_\ell) < (a', b')$. Altogether,

$$(a, b) < (a_k, b_0) < (a_{k+1}, b_\ell) < (a', b').$$

As a direct consequence of Eq. (4), we obtain

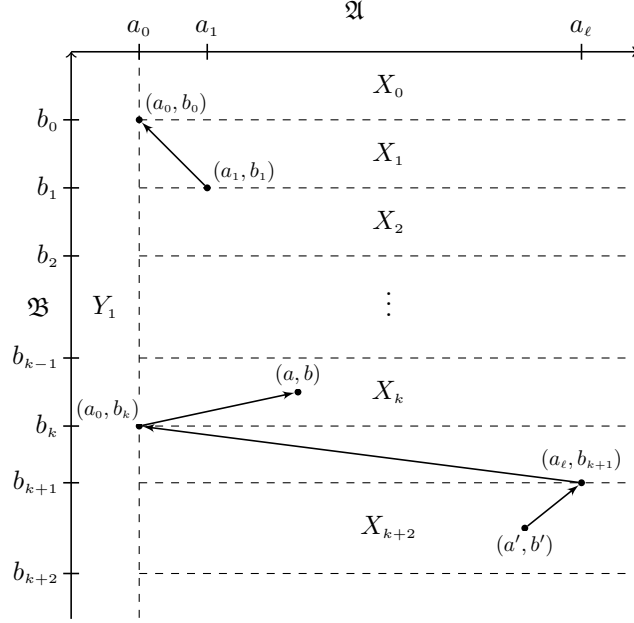
$$\mathfrak{A} \upharpoonright Y_2 = \sum_{k \in \omega} \mathfrak{A} \upharpoonright X_{2k} \quad \mathfrak{A} \upharpoonright Y_3 = \sum_{k \in \omega} \mathfrak{A} \upharpoonright X_{2k+1}.$$

Since every $\mathfrak{A} \upharpoonright X_{2k}$ is a finite sum of elements from $\mathcal{VD}_{<\alpha \oplus \beta}$, $\mathfrak{A} \upharpoonright Y_2$ is an ω -sum of elements from $\mathcal{VD}_{<\alpha \oplus \beta}$ and hence $\text{VD}_*(Y_2) \leq \alpha \oplus \beta$. Analogously, $\text{VD}_*(Y_3) \leq \alpha \oplus \beta$. This completes Case 1.1.

Case 1.2. $(a_0, b_0) > (a_1, b_1)$.

This case is very similar to Case 1.1 and depicted in Figure 2. To see this, let

$$X_0 = (a_0, \infty)_{\mathfrak{A}} \times [b_0, \infty)_{\mathfrak{B}} \quad X_k = (a_0, \infty)_{\mathfrak{A}} \times [b_i, b_{i-1})_{\mathfrak{B}} \text{ for } k \geq 1$$



■ **Figure 2** Proof sketch for Case 1.2.

and

$$Y_1 = (-\infty, a_0]_{\mathfrak{A}} \times B \qquad Y_2 = \bigcup_{k \in \mathbb{N}} X_{2k} \qquad Y_3 = \bigcup_{k \in \mathbb{N}} X_{2k+1}.$$

Again, we obtain $\text{VD}_*(X_k) < \alpha \oplus \beta$ for all $k \in \mathbb{N}$ as well as $\text{VD}_*(Y_1) < \alpha \oplus \beta$. Moreover, for each $k \in \mathbb{N}$ it holds that $X_k \gg X_{k+2}$ and hence

$$\mathfrak{A} \upharpoonright Y_2 = \sum_{k \in \omega^*} \mathfrak{A} \upharpoonright X_{2k} \qquad \mathfrak{A} \upharpoonright Y_3 = \sum_{k \in \omega^*} \mathfrak{A} \upharpoonright X_{2k+1}.$$

Consequently, $\text{VD}_*(Y_2), \text{VD}_*(Y_3) \leq \alpha \oplus \beta$. This completes Case 1.2 and hence Case 1.

Case 2. \mathfrak{A} and \mathfrak{B} both are ω -sums.

Consider the strictly increasing, unbounded sequences $(a_i)_{i \in \mathbb{N}}$ in \mathfrak{A} and $(b_i)_{i \in \mathbb{N}}$ in \mathfrak{B} which exist by Lemma 4.13. Depending on how (a_0, b_1) compares to (a_1, b_0) in \mathfrak{C} , we distinguish two cases.

Case 2.1. $(a_0, b_1) < (a_1, b_0)$.

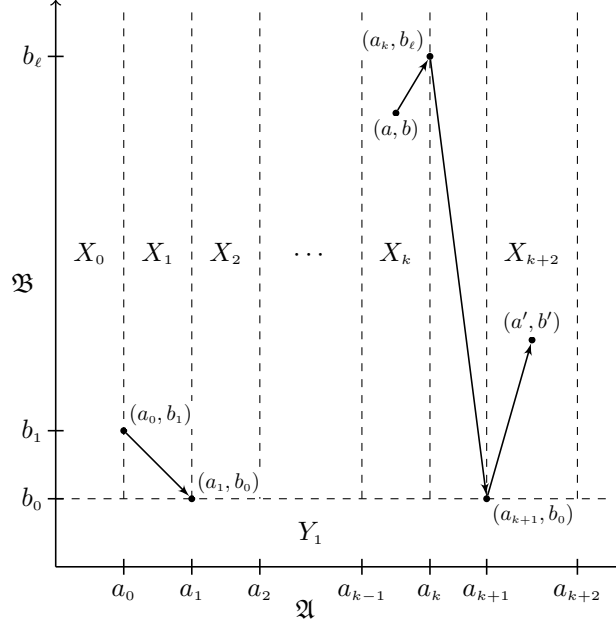
This case is treated similar to Case 1.1 and depicted in Figure 3.

Case 2.2. $(a_0, b_1) > (a_1, b_0)$.

This case is symmetric to Case 2.1.

Case 3. \mathfrak{A} is an ω^* -sum and \mathfrak{B} is an ω -sum.

This case is symmetric to Case 1.



■ **Figure 3** Proof sketch for Case 2.1.

Case 4. \mathfrak{A} and \mathfrak{B} both are ω^* -sums.

This case is dual to Case 2.

This finishes the proof of Proposition 4.15. ◀

Finally, we are in a position to perform the induction which proves Proposition 4.11.

Proof of Proposition 4.11. We show the claim by induction on n .

Base case. $n = 1$.

Clearly, $\mathfrak{A} \cong \mathfrak{B}_1$ and hence $\text{VD}_*(\mathfrak{A}) = \text{VD}_*(\mathfrak{B}_1)$.

Inductive step. $n > 1$.

To simplify notation, we assume that \mathfrak{A} is a linearisation of $\mathfrak{B}_1 \times \dots \times \mathfrak{B}_n$. For each i let $c_i: B_i^2 \rightarrow Q_i$ be a \leq -colouring of \mathfrak{B}_i such that

$$c: (B_1 \times \dots \times B_n)^2 \rightarrow Q_1 \times \dots \times Q_n, (\bar{a}, \bar{b}) \mapsto (c_1(a_1, b_1), \dots, c_n(a_n, b_n))$$

is a \leq -colouring of \mathfrak{A} . We consider the relation \sim on B_1 which is defined by $x \sim y$ iff $c_1(x, x) = c_1(y, y)$. This is an equivalence relation with at most $|Q_1|$ equivalence classes, say $X_1, \dots, X_m \subseteq B_1$ are these \sim -classes. Obviously, \mathfrak{A} is a sum augmentation of the m orderings $\mathfrak{A}|(X_i \times B_2 \times \dots \times B_n)$ for $i = 1, \dots, m$. By Proposition 4.9, it suffices to show for each i the inequality

$$\text{VD}_*(\mathfrak{A}|(X_i \times B_2 \times \dots \times B_n)) \leq \text{VD}_*(\mathfrak{B}_1) \oplus \dots \oplus \text{VD}_*(\mathfrak{B}_n). \quad (5)$$

Therefore, define for each $x \in B_1$ a scattered linear ordering \mathfrak{C}_x by $\|\mathfrak{C}_x\| = B_2 \times \dots \times B_n$ and $\bar{a} \leq^{\mathfrak{C}_x} \bar{b}$ iff $(x, \bar{a}) \leq^{\mathfrak{A}} (x, \bar{b})$. Clearly, \mathfrak{C}_x is a tame box augmentation of $\mathfrak{B}_2, \dots, \mathfrak{B}_n$ and

hence

$$\text{VD}_*(\mathfrak{C}_x) \leq \text{VD}_*(\mathfrak{B}_2) \oplus \cdots \oplus \text{VD}_*(\mathfrak{B}_n) \quad (6)$$

by induction. For $x, y \in B_1$ with $x \sim y$ and all $\bar{a}, \bar{b} \in B_2 \times \cdots \times B_n$ we have $c((x, \bar{a}), (x, \bar{b})) = c((y, \bar{a}), (y, \bar{b}))$ and hence $\bar{a} \leq^{\mathfrak{C}_x} \bar{b}$ iff $\bar{a} \leq^{\mathfrak{C}_y} \bar{b}$, i.e., $\mathfrak{C}_x = \mathfrak{C}_y$. For any \sim -class $X_i \subseteq B_1$ and every $x \in X_i$ we obtain that $\mathfrak{A} \upharpoonright (X_i \times B_2 \times \cdots \times B_n)$ is a tame box augmentation of $\mathfrak{B}_1 \upharpoonright X_i$ and \mathfrak{C}_x . Finally, Eq. (5) follows from $\text{VD}_*(\mathfrak{B}_1 \upharpoonright X_i) \leq \text{VD}_*(\mathfrak{B}_1)$, Eq. (6), and Proposition 4.15. \blacktriangleleft

4.4 Proof of the Main Result

In order to conclude Theorem 4.4 from Corollaries 3.7, 4.10, and 4.12, we need another auxiliary result. Statement (1) of the lemma below is in fact shown by the proof of Proposition 4.5 in [7].

► **Lemma 4.16.** *Let \mathfrak{A} be a linear ordering and $\alpha < \text{FC}(\mathfrak{A})$.*

1. *\mathfrak{A} contains a scattered closed interval I with $\text{FC}(I) = \alpha + 1$.*
2. *\mathfrak{A} contains a scattered closed interval I with $\text{VD}_*(I) = \alpha$.*

Proof. We only show (2). By (1), there exists a closed scattered interval I of \mathfrak{A} with $\text{VD}(I) = \text{FC}(I) = \alpha + 1$. Since I has a least and a greatest element, it is neither an ω -sum nor an ω^* -sum nor a ζ -sum of elements from $\mathcal{VD}_{<\alpha+1} = \mathcal{VD}_\alpha$. Thus, I is a finite sum of elements from \mathcal{VD}_α and hence $\text{VD}_*(I) \leq \alpha$. Due to Eq. (2), $\text{VD}_*(I) = \alpha$. \blacktriangleleft

Now, we are prepared to provide the missing proof of the main result.

Proof of Theorem 4.4. By contradiction, assume there exists a tree-automatic linear ordering \mathfrak{A} with $\text{FC}(\mathfrak{A}) \geq \omega^\omega$. Consider the formula $\phi(x, y_1, y_2) = y_1 \leq x \wedge x \leq y_2$. By Lemma 4.16, for each $d \in \mathbb{N}$ there exists a scattered closed interval $I = [b_1, b_2]_\mathfrak{A}$ in \mathfrak{A} with $b_1 \leq b_2$ and $\text{VD}_*(I) = \omega^d$. Since $I = \phi^\mathfrak{A}(\cdot, b_1, b_2)$ and ω^d is VD_* -sum-indecomposable as well as VD_* -tame-box-indecomposable, this contradicts Corollary 3.7. \blacktriangleleft

5 \mathfrak{T}_2 -Free Tree-Automatic Presentations

In this section, we investigate a restricted form of tree-automaticity where only those tree-automatic presentations $(\mathcal{A}; (\mathcal{A}_R)_{R \in \mathcal{R}})$ are permitted for which the binary tree

$$T(\mathcal{A}) = T(L(\mathcal{A})) = \bigcup_{t \in L(\mathcal{A})} \text{dom}(t)$$

is of bounded branching complexity—in some sense defined later.³ The main result of this section, namely Theorem 5.4, states that any linear ordering \mathfrak{A} which admits a tree-automatic presentation whose branching complexity is bounded by $k \in \mathbb{N}$ satisfies $\text{FC}(\mathfrak{A}) < \omega^k$.

³ Roughly speaking, the branching complexity is bounded if the infinite full binary tree cannot be embedded and is measured in terms of the Cantor-Bendixson rank.

5.1 Binary Trees and the Cantor-Bendixson Rank

The *infinite full binary tree* is the set $\mathfrak{T}_2 = \{0, 1\}^*$ whose nodes are ordered by the prefix-relation \preceq . A *binary tree* is a (possibly empty) prefix-closed subset $T \subseteq \mathfrak{T}_2$. The (isomorphism type of) the *subtree* rooted at $u \in T$ is

$$T \upharpoonright u = \{ v \in \{0, 1\}^* \mid uv \in T \}.$$

A binary tree T is *regular* if it is a regular language. Due to the Myhill-Nerode theorem, this is equivalent to the fact that T has (up to isomorphism) only finitely many distinct subtrees $T \upharpoonright u$. To every tree language $L \subseteq T_\Sigma$ we assign a binary tree

$$T(L) = \bigcup_{t \in L} \text{dom}(t).$$

► **Lemma 5.1.** *For every regular tree language $L \subseteq T_\Sigma$ the binary tree $T(L)$ is regular.*

Proof. Let \mathcal{A} be a tree automaton recognising L . For each $u \in T(L)$ let

$$Q(u) = \{ \mathcal{A}(t, u) \mid t \in L \}.$$

It is easy to see that $Q(u) = Q(v)$ implies $T(L) \upharpoonright u = T(L) \upharpoonright v$. Thus, $T(L)$ is regular. ◀

A binary tree T is called *\mathfrak{T}_2 -free* if \mathfrak{T}_2 cannot be embedded into T , i.e., there is no injection $f: \mathfrak{T}_2 \rightarrow T$ such that $u \preceq v$ iff $f(u) \preceq f(v)$ for all $u, v \in \mathfrak{T}_2$. An *infinite branch* of a binary tree T is an infinite subset $P \subseteq T$ which is prefix-closed and linearly ordered by \preceq . The *derivative* of T is the set $d(T)$ of all $u \in T$ which are contained in at least two distinct infinite branches of T . Clearly, $d(T)$ is a binary tree. For $n \in \mathbb{N}$ let $d^{(n)}(T)$ be the n^{th} derivation of T , i.e., $d^{(0)}(T) = T$ and $d^{(n)}(T) = d(d^{(n-1)}(T))$ for $n > 0$. Whenever T is regular there exists an $n \in \mathbb{N}$ such that $d^{(n)}(T) = d^{(k)}(T)$ for all $k \geq n$ and $d^{(n)}(T)$ is finite precisely if T is \mathfrak{T}_2 -free [7].

► **Definition 5.2.** Let T be a regular, \mathfrak{T}_2 -free binary tree. The *CB_{*}-rank* of T , denoted by $\text{CB}_*(T)$, is the least $n \in \mathbb{N}$ such that $d^{(n)}(T)$ is finite.⁴

Clearly, $d(T \upharpoonright u) = d(T) \upharpoonright u$ and hence $\text{CB}_*(T \upharpoonright u) \leq \text{CB}_*(T)$ for all $u \in T$.

► **Definition 5.3.** A tree-automatic presentation $(\mathcal{A}; (\mathcal{A}_R)_{R \in \mathcal{R}})$ is *\mathfrak{T}_2 -free* if $T(L(\mathcal{A}))$ is \mathfrak{T}_2 -free and then its *rank* is the CB_* -rank of $T(L(\mathcal{A}))$.⁵

► **Remark.** Obviously, the structures which admit a \mathfrak{T}_2 -free tree-automatic presentation of rank 0 are precisely the finite structures. Furthermore, it can be shown that the structures which admit a presentation of rank at most 1 are exactly the string-automatic structures.⁶

⁴ In fact, CB_* is a variation of the Cantor-Bendixson rank which was adapted to trees in [7].

⁵ In [1] the authors speak of *bounded-rank tree-automatic presentations*. Their notion of *rank* is defined differently, but can be shown to be equivalent to ours.

⁶ String-automatic structures are defined like tree-automatic structures but with finite words and finite automata instead of trees and tree automata.

5.2 \mathfrak{T}_2 -Free Tree-Automatic Presentation of Linear Orderings

The following is the main result of this section.

► **Theorem 5.4.** *Let \mathfrak{A} be a linear ordering which admits a \mathfrak{T}_2 -free tree-automatic presentation of rank $k \geq 1$. Then*

$$\text{FC}(\mathfrak{A}) < \omega^k.$$

► **Corollary 5.5.** *An ordinal α admits a \mathfrak{T}_2 -free tree-automatic presentation of rank at most k if, and only if,*

$$\alpha < \omega^{\omega^k}.$$

► **Remark.** As direct consequence of this corollary and Corollary 3.11, every tree-automatic ordinal already admits a \mathfrak{T}_2 -free tree-automatic presentation. In fact, Jain, Khoushainov, Schlicht, and Stephan [6] recently showed that every tree-automatic presentation of an ordinal—or more generally, of a scattered linear ordering—is \mathfrak{T}_2 -free.

The proof of Theorem 5.4 works by more detailed inspection of the proofs of Theorem 3.6, Corollary 3.7, and Theorem 4.4 in combination with the following lemma.

► **Lemma 5.6.** *Let T be a regular, \mathfrak{T}_2 -free binary tree. Then there exists a constant $C \in \mathbb{N}$ such that any anti-chain $A \subseteq T$ contains at most C elements u with $\text{CB}_*(T \upharpoonright u) = \text{CB}_*(T)$.*

Proof. If $\text{CB}_*(T) = 0$ then T is finite and the claim is trivially satisfied. Thus, assume $\text{CB}_*(T) = k > 0$. Let $n \in \mathbb{N}$ be the *index* of T , i.e., the size of the set $\{T \upharpoonright u \mid u \in T\}$. We show that $C = 2^n$ is a possible choice.

By contradiction, suppose there is an anti-chain A consisting of $2^n + 1$ elements $u \in T$ satisfying $\text{CB}_*(T \upharpoonright u) = k$. Let B be the set of all $v \in T$ which are the longest common prefix of two distinct elements from A . Then B contains exactly 2^n elements. For every $u \in A$ the set $d^{(k-1)}(T \upharpoonright u) = d^{(k-1)}(T) \upharpoonright u$ is infinite. By König's lemma, there exists an infinite branch of $d^{(k-1)}(T)$ containing u . Thus, $B \subseteq d^{(k)}(T)$. For every $v \in d^{(k)}(T)$ it holds that $d^{(k)}(T) \upharpoonright v = d^{(k)}(T \upharpoonright v)$ and hence the index of $d^{(k)}(T)$ is at most n . Since $d^{(k)}(T)$ contains at least 2^n elements, a simple pumping argument shows that $d^{(k)}(T)$ is infinite. But this contradicts $\text{CB}_*(T) = k$. ◀

Now, we are in a position to show the main result of this section.

Proof of Theorem 5.4. We show the claim by induction on $k \geq 1$. Therein, we use the induction hypothesis only in the following restricted form: Every scattered linear ordering \mathfrak{A} which admits a \mathfrak{T}_2 -free tree-automatic presentation of rank $k \geq 0$ satisfies $\text{VD}_*(\mathfrak{A}) < \omega^k$. For $k \geq 1$ this assertion easily follows from $\text{VD}(\mathfrak{A}) = \text{FC}(\mathfrak{A}) < \omega^k$.

Base case. $k = 0$.

Since any structure which admits a \mathfrak{T}_2 -free tree-automatic presentation of rank 0 is finite, every such scattered linear ordering \mathfrak{A} trivially satisfies $\text{VD}_*(\mathfrak{A}) = 0 < \omega^0$.

Inductive step. $k \geq 1$.

By contradiction, assume there exists a tree-automatic linear ordering \mathfrak{A} which admits a \mathfrak{T}_2 -free tree-automatic presentation $(\mathcal{A}; (\mathcal{A}_R)_{R \in \mathcal{R}})$ of rank k and satisfies $\text{FC}(\mathfrak{A}) \geq \omega^k$. To keep notation simple, we assume that the naming function $\mu: A \rightarrow L(\mathcal{A})$ is the identity, i.e., \mathfrak{A} is identified with its tree-automatic copy $\mu(\mathcal{A})$. Let C be the constant which exists by

Lemma 5.6 for the binary tree $T(A)$. Moreover, let $\mathcal{S}_\phi^{\mathfrak{A}}$ be the set which is constructed in the proof of Theorem 3.6 from $\bar{\mathcal{A}}$ and the formula $\phi(x, y_1, y_2) = y_1 \leq x \wedge x \leq y_2$. We show that $\mathcal{S}_\phi^{\mathfrak{A}}$ contains for each $n \in \mathbb{N}$ a scattered linear ordering \mathfrak{B} with $\omega^{k-1}n < \text{VD}_*(\mathfrak{B}) < \omega^k$. This contradicts the finiteness of $\mathcal{S}_\phi^{\mathfrak{A}}$ and proves the theorem.

Therefore, consider some $n \in \mathbb{N}$. By Lemma 4.16, there exists a scattered closed interval $I = [a_1, a_2]_{\mathfrak{A}}$ of \mathfrak{A} with $a_1 \leq a_2$ and $\text{VD}_*(I) = \omega^{k-1}(nC + 1)$. Now, we delve into the details of the proof of Theorem 3.6. Since $I = \phi^{\mathfrak{A}}(\cdot, a_1, a_2)$ and $\omega^{k-1}(nC + 1)$ is VD_* -sum-indecomposable, there exists a $\sim_{(a_1, a_2)}$ -class $B \subseteq I$ such that $\text{VD}_*(B) = \omega^{k-1}(nC + 1)$. Let $\vartheta = (t_D, U, (\rho_R)_{R \in \{\phi\} \uplus \mathcal{R}})$ be the corresponding (a_1, a_2) -type, u_1, \dots, u_r an enumeration of U , and $\mathfrak{S}_i = \mathfrak{S}_{\gamma(\vartheta, u_i)}$ for each $i = 1, \dots, r$. Notice that the \mathfrak{S}_i are scattered linear orderings and form a tame box decomposition of $\mathfrak{A} \upharpoonright B$. It is easy to see that $T(S_i) \subseteq T(A) \upharpoonright u_i$ and hence $\text{CB}_*(T(S_i)) \leq k$ for each i . Since U is an anti-chain in $T(A)$, equality holds true in at most C cases. Without loss of generality, there exists a $p \leq C$ such that $\text{CB}_*(T(S_i)) = k$ for $i \leq p$ and $\text{CB}_*(T(S_i)) < k$ for $i > p$.

By the restricted induction hypothesis, we obtain $\text{VD}_*(\mathfrak{S}_i) < \omega^{k-1}$ for $i > p$. If we had $\text{VD}_*(\mathfrak{S}_i) \leq \omega^{k-1}n$ for each $i = 1, \dots, p$, then

$$\underbrace{\text{VD}_*(\mathfrak{S}_1) \oplus \dots \oplus \text{VD}_*(\mathfrak{S}_p)}_{\leq \omega^{k-1}np} \oplus \underbrace{\text{VD}_*(\mathfrak{S}_{p-1}) \oplus \dots \oplus \text{VD}_*(\mathfrak{S}_r)}_{< \omega^{k-1}} < \omega^{k-1}(nC + 1).$$

This would contradict Proposition 4.11 and hence there exists a $j \in \{1, \dots, p\}$ with $\text{VD}_*(\mathfrak{S}_j) > \omega^{k-1}n$. Since \mathfrak{S}_j can be embedded into $\mathfrak{A} \upharpoonright B$, we further obtain

$$\text{VD}_*(\mathfrak{S}_j) \leq \omega^{k-1}(nC + 1) < \omega^k. \quad \blacktriangleleft$$

In order to verify Corollary 5.5 we still have to prove that every ordinal $\alpha < \omega^{\omega^k}$ admits a \mathfrak{T}_2 -free tree-automatic presentation of rank at most k .

Proof of Corollary 5.5. The “only if”-part follows directly from Theorem 5.4 and we only need to show the “if”-part. For $k = 0$ the claim is trivial since each ordinal $\alpha < \omega$ is finite. Thus, assume $k > 0$ and consider some $\alpha < \omega^{\omega^k}$. There exists an $n \in \mathbb{N}$ such that $\alpha < \omega^{\omega^{k-1}n}$. The ordinal $\omega^{\omega^{k-1}n}$ can be regarded as the lexicographically ordered set of all n -tuples of elements from $\omega^{\omega^{k-1}}$. Let $\bar{\mathcal{A}}$ be the tree-automatic presentation of $\omega^{\omega^{k-1}}$ which was constructed in Lemma 3.10 and $\nu: A \rightarrow T_\Sigma$ the corresponding naming function. A closer look at the induction in the proof of Lemma 3.10 reveals that $\bar{\mathcal{A}}$ is \mathfrak{T}_2 -free and of rank k . The map $\mu: \omega^{\omega^{k-1}n} \rightarrow T_{\Sigma^n}$ with

$$\mu(\beta_1, \dots, \beta_n) = \otimes(\nu(\beta_1), \dots, \nu(\beta_n))$$

can be used as naming function for a \mathfrak{T}_2 -free tree-automatic presentation of rank k of $\omega^{\omega^{k-1}n}$. Finally, α is FO-definable with one parameter in $\omega^{\omega^{k-1}n}$ and hence admits a \mathfrak{T}_2 -free tree-automatic presentation of rank k as well. \blacktriangleleft

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